ISRAEL JOURNAL OF MATHEMATICS, Vol. 69, No. 3, 1990

ON AMENABLE GROUPS OF AUTOMORPHISMS ON VON NEUMANN ALGEBRAS†

BY

EDMOND E. GRANIRER

Department of Mathematics, The University of British Columbia, Vancouver, B.C., Canada

ABSTRACT

Let $G \subset \operatorname{Aut} \mathcal{M}$ be a countable group, $\mathcal{M}$ a Von Neumann algebra. Let E be a set of pure states on $\mathcal{M}$ such that $G^*E \subset E$, S^G be the set of G invariant states on $\mathcal{M}$ and $S_E^G = S^G \cap w^* \operatorname{cl} \operatorname{co} E$. We investigate in this paper some geometric properties for the set S_E^G which turn out to be equivalent to amenability for the group G. For example, we show that $S_E^G \subset \mathcal{M}_*$ (S_E^G has the WRNP) implies that $\mathcal{M}$ contains minimal projections ($\hat{E}$ contains *finite* G invariant orbits) hold true, for all $\mathcal{M}$ iff G is amenable. Furthermore we show that if G is amenable then $S^G \cap \mathcal{M}_*^\perp$ contains a big set, thus improving results obtained by Ching Chou in [2]. These results imply that no action of an amenable countable group G on an arbitrary W^* algebra $\mathcal{M}$ is s — strongly ergodic. Moreover card $S^G \cap \mathcal{M}_*^\perp \geqq 2^c$ (see M. Choda [4], K. Schmidt [21] and compare with A. Connes and B. Weiss [5]).

Introduction

Let $\mathcal{M}$ be an infinite-dimensional W^* algebra, $\mathcal{M}^*$ ($\mathcal{M}_*$) its dual (predual). Let Aut $\mathcal{M}$ be the group of all automorphisms of $\mathcal{M}$ onto $\mathcal{M}$ and $G \subset \operatorname{Aut} \mathcal{M}$ a group. Let $E \subset \mathcal{M}^*$ be a set of pure states (as in [23]) (usually) such that $G^*E \subset E$ (i.e. $g^*E \subset E$ for all g in G, where $\langle g^*\psi, x \rangle = \langle \psi, gx \rangle$ if $\psi \in \mathcal{M}^*$, $x \in \mathcal{M}$).

Denote by S^G the set of all states ψ in $\mathcal{M}^*$ such that $G^*\psi = \psi$ (i.e. $g^*\psi = \psi$ for all g in G). Let $S_E^G = S^G \cap w^* \operatorname{cl} \operatorname{co} E$ where $w^* \operatorname{cl}$ denotes w^* closure and co E the convex hull of E. If $\mathcal{J} \subset \mathcal{M}$, let $\mathcal{J}^0 = \{\psi \in \mathcal{M}^*; \langle \psi, x \rangle = 0 \text{ for all } x \text{ in } \mathcal{J}\}$.

† The author gratefully acknowledges the support of an Izaak Walton Killam Memorial Senior Fellowship.

Received October 27, 1988 and in revised form October 2, 1989

If E is the set of all pure states on $\mathcal{M}$, then clearly $S_E^G = S^G$, while if $\mathcal{J} = \{0\}$ then $S_E^G \cap \mathcal{J}^0 = S_E^G$.

Our first result is a characterization of amenability for G in terms of minimal projections in $\mathcal{M}$, namely:

THEOREM 1. *Let $G \subset \operatorname{Aut} \mathcal{M}$ be a group, $E \subset \mathcal{M}^*$ a set of pure states such that $G^*E \subset E$ and $\mathcal{J}$ countable. Assume that $\varnothing \neq S_E^G \cap \mathcal{J}^0 \subset \mathcal{M}_*$.*

(a) *If G is countable and* amenable *then $\mathcal{M}$ contains minimal projections.*

(b) *If G is not amenable then the action of G on $\mathcal{M} = L^\infty(T^G, \lambda^G)$ is such that $S^G \subset \mathcal{M}_*$, yet $\mathcal{M}$ does not contain minimal projections.*

Part (b) of the theorem is an immediate consequence of the following interesting result of Losert and Rindler [15] and, independently, of Rosenblatt [17]: "If G is any *nonamenable* group, then the ergodic measure preserving action of G on $\mathcal{M} = L^\infty(T^G, \lambda^G)$ admits a unique invariant state, i.e. $S^G = \{\lambda^G\}$". Here $T^G = \{\Pi\, T_g; g \in G\}$ is the product group of $T_g = T$ (the torus) for each g in G, equipped with its Haar measure λ^G.

The next result is concerned with the "bigness" of the set S^G. Let

$$c_0 = \left\{ b = (b_k) \in l^\infty;\ \lim_k b_k = 0 \right\}$$

and $\mathcal{F}$ be the set of states ψ on l^∞ such that $\psi = 0$ on c_0 (i.e. $\langle \psi, \delta_n \rangle = 0$ for all n where δ_n is defined as zero at all $k \neq n$ and is one at $k = n$). Thus $\beta N \sim N \subset \mathcal{F}$ and card $\beta N \sim N = 2^c$, where $c = \operatorname{card} R$ is the cardinality of the continuum. Furthermore $\beta N \sim N$ is a w^* perfect set such that if $\psi_1, \psi_2 \in \beta N \sim N$ and $\psi_1 \neq \psi_2$ then $\| \psi_1 - \psi_2 \| = 2$. In particular $\mathcal{F}$ is as "big" as it can be, since $\operatorname{card}(l^{\infty *}) = 2^c$.

Denote by $\mathcal{M}^* \sim \mathcal{M}_*$ the set theoretical difference between $\mathcal{M}^*$ and $\mathcal{M}_* \subset \mathcal{M}^*$. We now have

THEOREM 2. *Let $\mathcal{M}$ be an infinite-dimensional σ-finite W^* algebra* (see [23] p. 78) *which contains no minimal projections. Let $G \subset \operatorname{Aut} \mathcal{M}$ be a countable amenable group and E a set of pure states on $\mathcal{M}$ such that $G^*E \subset E$. If for some countable $\mathcal{J} \subset \mathcal{M}$, $S_E^G \cap \mathcal{J}^0 \neq \varnothing$ ($\mathcal{J} = \{0\}$ is such), then $S^G \cap \mathcal{J}^0 \cap (\mathcal{M}^* \sim \mathcal{M}_*)$ contains a w^* and norm isomorphic copy of the big set $\mathcal{F}$.*

If G is not amenable, then the action of G on $\mathcal{M} = L^\infty(T^G, \lambda^G)$ is such that $S^G = \{\lambda^G\}$, thus card $S^G = 1$.

We note that the set $S_E^G \cap (\mathcal{M}^* \sim \mathcal{M}_*)$ need not be big. Even if $G = \{e\}$ is the trivial group, $S_E^G \cap (\mathcal{M}^* \sim \mathcal{M}_*)$ may contain just one element. Nevertheless, by the above theorem $S^G \cap (\mathcal{M}^* \sim \mathcal{M}_*)$ has to be "big". Furthermore

$S^G \cap \mathcal{J}^0 \cap (\mathcal{M}^* \sim \mathcal{M}_*)$ may contain $\mathcal{F}$, yet $S^G \cap \mathcal{J}^0 \cap \mathcal{M}_*^\perp = \emptyset$ may happen. ($\mathcal{M}_*^\perp$ is the set of singular elements of $\mathcal{M}^*$ as in Takesaki [23] p. 127.)

Why are we interested in $S_E^G \cap \mathcal{J}^0$ where $\mathcal{J} \subset \mathcal{M}$ is countable, instead of only in S_E^G? It happens that one is interested in a set of states which are G invariant yet are *not* invariant with respect to some maps $h_n: \mathcal{M} \to \mathcal{M}$, $n = 1, 2, 3, \ldots$ (a "symmetry breaking" situation as in [18]). In such a case there exist some $\phi_0 \in S_E^G$, $x_n \in \mathcal{M}$ and scalars α_n, β_n such that $\phi_0(x_0) = \alpha_n$, $\phi_0(h_n x_n) = \beta_n$ for all n. Hence, if we take $\mathcal{J} = \{x_n - \alpha_n I\} \cup \{h_n x_n - \beta_n I\}$, then $\phi_0 \in S_E^G \cap \mathcal{J}^0$. One may then be interested to find out when $S_E^G \cap \mathcal{J}^0$ has the WRNP (see Theorem 3) or when $S^G \cap \mathcal{J}^0$ contains $\mathcal{F}$ (as in Theorems 2, 7 and 9).

The following is a result of Ching Chou:

COROLLARY (Ching Chou [2]). *Assume that G is a countable amenable group acting ergodically as measure preserving maps on a nonatomic probability space $(X, \mathcal{B}, p)$. Then there exists a positive w^*–w^* continuous isometry into, $t^*: l^{\infty *} \to L^\infty(X)^*$ such that $t^* \mathcal{F} \subset S^G$.*

We improve this result in

THEOREM 7. *Let $\mathcal{M}$ be an infinite-dimensional W^* algebra, $G \subset \operatorname{Aut} \mathcal{M}$ a countble amenable group. Then there exists a sequence of σ-finite projections $\{q_n; n \geqq 0\}$ such that $q_n \uparrow q_0$ σ-strongly, $gq_0 = q_0$ for all g in G and a positive w^*–w^* continuous isometry into $t^*: l^{\infty *} \to \mathcal{M}^*$ such that $t^*(\mathcal{F}) \subset S^G \cap \mathcal{M}_*^\perp \cap P^0$ where $P = \{q_n; n \geqq 1\} \cup \{I - q_0\}$.*

The main result of Ching Chou in [2] is the

THEOREM. *Let G be a countable group acting ergodically as measure preserving maps on the nonatomic probability space $(X, \mathcal{B}, p)$. If the set S^G of G invariant states on $\mathcal{M} = L^\infty(X, p)$ contains some state $\psi_0 \neq p$ (thus $\psi_0 \notin L^1(X, p)$) then there exists a positive w^*–w^* continuous isometry into $t^*: l^{\infty *} \to \mathcal{M}^*$ such that $t^* \mathcal{F} \subset S^G$.*

We improve Chou's result in

THEOREM 9. *Let $G \subset \operatorname{Aut} \mathcal{M}$ be any countable group acting on the σ-finite, infinite-dimensional, W^* algebra $\mathcal{M}$. Assume that there is some ϕ_0 in S^G which is not normal. Then there is a sequence of projetions $q_n \uparrow I$, σ-strongly, and a positive w^*–w^* continuous isometry $t^*: l^{\infty *} \to \mathcal{M}^*$ such that $t^* \mathcal{F} \subset S^G \cap \mathcal{M}_*^\perp \cap P^0$ where $P = \{q_n\}$.*

In Chou's proofs essential use is made of the fact that G preserves the finite measure p, thus p is a finite G invariant faithful trace on $\mathcal{M} = L^\infty(X, p)$. In our case $\mathcal{M}$ need not admit even a semifinite faithful trace.

This paper contains results related to the theorem announced on p. 760 of our paper [13].

We bring hereby a short *errata* to [13]:

On p. 755^{15} replace $(X, \mathcal{B}, \mu)$ by $(\mathcal{X}, \mathcal{B}, \mu)$. On p. 755_{19} and 755_{20} replace S by G. On p. 758 footnote to Corollary 2: "Using recent results of N. Ghoussoub, G. Godefroy, B. Maurey and W. Schachermeyer one can replace RNP by WRNP in this corollary. Corollary 4 becomes thus superfluous." On p. 758^{22} replace S^G_E by $S^G_{E_1}$. On p. 760_3 replace "is" by "improves".

Definitions and notations

Throughout this paper $\mathcal{M}$ will denote an arbitrary W^* algebra, $\mathcal{M}^*$ ($\mathcal{M}_*$) its dual (predual) and $I \in \mathcal{M}$ the identity. Then, as is well known (see Takesaki [23] p. 127), $\mathcal{M}^* = \mathcal{M}_* \oplus \mathcal{M}_*^\perp$, an l_1 direct sum, where $\mathcal{M}_*^\perp$ denotes the set of singular functionals on $\mathcal{M}$. $\mathcal{M}^* \sim \mathcal{M}_*$ will denote the set theoretical difference of $\mathcal{M}^*$ and $\mathcal{M}_*$. A self-adjoint projection in $\mathcal{M}$ will be called a projection. If $\phi \in \mathcal{M}^*$ is positive then $(\pi_\phi, H_\phi, \xi_\phi)$ will denote the GNS representation induced by ϕ as in [23] p. 39 Theorem I.9.14. $B(H)$ is the W^* algebra of all bounded operators on the Hilbert space H.

Let $G \subset \operatorname{Aut} \mathcal{M}$ be a group, E a set of pure states on $\mathcal{M}$ (usually) such that $G^*E \subset E$. Let $S^G_E = \{\psi \in w^* \operatorname{cl} \operatorname{co} E; G^*\psi = \psi\}$ and, if E is the set of all pure states, denote S^G_E by S^G. If $A \subset \mathcal{M}^*$, $w^*G_\delta(A)$ denotes the w^*G_δ points of A. If $A \subset \mathcal{M}^*$ is w^* compact then $x_0 \in w^*G_\delta(A)$, iff for some countable $\mathcal{J} \subset \mathcal{M}$, $\{x_0\} = A \cap \mathcal{J}^0$. Then $x_0 \in w^*G_\delta(A)$ iff the w^* topology restricted to A is first countable at x_0.

If $L^\infty = L^\infty(X, \mu)$, Δ_{L^∞} denotes the set of all multiplicative (i.e. pure) states on L^∞. In all the theorems where the isometry (isomorphism) $t^* : l^{\infty *} \to \mathcal{M}^*$ appears, t^* is the adjoint of an onto positive operator $t : \mathcal{M} \to l^\infty$ as defined in the proof of Theorem 5 or in Theorem 1.4 of [12].

A convex set K of a Banach space X has the (WRNP) RNP if, for any finite measure space $(\mathcal{X}, B, \mu)$, any countably additive μ-continuous map $m : B \to X$ such that $\mu(A)^{-1} m(A) \in K$ if $\mu(A) \neq 0$ is represented by a Bochner (Pettis) integrable function. For very interesting geometric properties of RNP (WRNP) sets see Stegall [22] (E. Saab [19]). δ_n will always denote the sequence $\delta_n = \{\delta^n_k\}$ where $\delta^n_k = 0$ if $k \neq n$ and $\delta^n_n = 1$.

The term "σ-finite" for a W^* algebra (projection) is the same as "countably decomposable" in Kadison–Ringrose [14]. If $K \subset \mathcal{M}^*$, w^* seq cl K denotes the w^* sequential closure of K.

I. Amenability, minimal projections and WRNP

THEOREM 1. *Let $G \subset \operatorname{Aut} \mathcal{M}$ be a countable group, E a set of pure states on the W^* algebra $\mathcal{M}$ such that $G^*E \subset E$ and $\mathcal{J} \in \mathcal{M}$ is countable. If G is amenable and if $\emptyset \neq S_E^G \cap \mathcal{J}^0 \subset \mathcal{M}_*$, then $\mathcal{M}$ contains minimal projections.*

If G is not amenable, then the action of G on $L^\infty(T^G, \lambda^G)$ with $E = \Delta_{L^\infty}$ and $\mathcal{J} = \{0\}$ satisfies $S^G \cap \mathcal{J}^0 = \{\lambda^G\} \subset \mathcal{M}_*$ yet $\mathcal{M}$ contains no minimal projections.

PROOF. Assume that $S_E^G \cap \mathcal{J}^0 \subset \mathcal{M}_*$. Then $S_E^G \cap \mathcal{J}^0$ is a weakly, i.e. $\sigma(\mathcal{M}_*, \mathcal{M})$ compact set (since it is w^* compact in $\mathcal{M}^*$). In particular $S_E^G \cap \mathcal{J}^0$ has the RNP (by Stegall [22] p. 508). Hence in any case $S_E^G \cap \mathcal{J}^0$ is a w^* compact convex RNP set and as such it has some w^*G_δ point ϕ_0 (by [22]; for an alternate proof see [11] p. 116). Applying our Corollary 1(a) of [13] we get that there is a countable subset $E_0 \subset E$ such that

$$(*) \qquad \pi_\psi \leqq \pi_{\phi_0} \leqq \{\oplus \pi_\phi ; \phi \in E_0\} \qquad \text{for each } \psi \in E_0.$$

However $(\pi_{\phi_0}, H_{\phi_0}, \xi_{\phi_0})$ is a normal morphism of $\mathcal{M}$ (since ϕ_0 is normal); see Pedersen [16] 3.3.9. Hence $\pi_{\phi_0}(\mathcal{M})$ is a W^* algebra, by [16] 2.5.3, and $\pi_{\phi_0}(\mathcal{M})$ is isomorphic to $\mathcal{M}q$ for some projection $q \in \mathcal{M} \cap \mathcal{M}'$, by [16] 2.5.5. If now $\psi \in E_0$, then by $(*)$

$$\psi(x) = \langle \pi_\psi(x)\xi_\psi, \xi_\psi \rangle = \langle \pi_{\phi_0}(x)\xi_1, \xi_1 \rangle, \qquad \text{for some } \xi_1 \in H_{\phi_0}.$$

Since π_{ϕ_0} is normal, it follows that ψ is normal. Hence $\pi_\psi(\mathcal{M})$ is also a W^* algebra. However, the W^* algebra generated by the C^* algebra $\pi_\psi(\mathcal{M})$ is $B(H_\psi)$ since π_ψ is irreducible. Hence $\pi_\psi(\mathcal{M}) = B(H_\psi)$ and, as above, $\pi_\psi(\mathcal{M})$ is isomorphic to a W^* subalgebra $\mathcal{M}q_1$ *of* $\mathcal{M}$. Since $B(H_\psi)$ contains minimal projections, so does $\mathcal{M}$. This proves the first part. For the second part, we only need to note that (T^G, λ^G) contains no atoms hence $\mathcal{M}$ has no minimal projections (see Introduction). ■

THEOREM 2. *Let $\mathcal{M}$ be a σ-finite W^* algebra and contain no minimal projections, $G \subset \operatorname{Aut} \mathcal{M}$ be countable amenable, E a set of pure states such $G^*E \subset E$. Then for every w^*G_δ set $U \subset \mathcal{M}^*$ such that $U \cap S_E^G \neq \emptyset$ (or any $U = \mathcal{J}^0$ for some countable $\mathcal{J} \subset \mathcal{M}$ with $S_E^G \cap U \neq \emptyset$) satisfies*

$S_E^G \cap U \cap \{\mathcal{M}^* \sim \mathcal{M}_*\} \neq \emptyset$ *and furthermore* $S^G \cap U \cap \{\mathcal{M}^* \sim \mathcal{M}_*\}$ *contains a* w^* *and norm isomorphic copy of the big set* $\mathcal{F}$ (see end of proof).

If G is not amenable, the action of G on $L^\infty(T^G, \lambda^G)$ *satisfies for* $E = \Delta_{L^\infty}$, $\mathcal{J} = \{0\}$, $S_E^G \cap \mathcal{J}^0 = S^G = \{\lambda^G\} \subset \mathcal{M}_*$.

Remark. The set $S_E^G \cap \mathcal{J}^0 \cap \{\mathcal{M}^* \sim \mathcal{M}_*\}$ *need not be big*. It may contain just one element. For example, if $G = \{e\}$ is the trivial group acting on $\mathcal{M} = L^\infty[0, 1]$ and if E contains only one pure (i.e. multiplicative) state $\{\phi_0\}$ and $\mathcal{J} = \{0\}$, then $S_E^G \cap \mathcal{J}^0 \cap \{\mathcal{M}^* \sim \mathcal{M}_*\} = \{\phi_0\}$. However $S^G \cap \mathcal{J}^0 \cap \{\mathcal{M}^* \sim \mathcal{M}_*\}$ is big.

Proof. Let $\phi_1 \in U \cap S_E^G$, then there are w^* open sets U_n such that $\phi_1 \in \bigcap_{n=1}^\infty U_n \cap S_E^G = U$. For each n there is a w^* neighbourhood

$$V_n = \{\phi; |(\phi - \phi_1)(x_k^n)| < \varepsilon_n, 1 \leqq k \leqq k_n\}$$

such that $V_n \cap S_E^G \subset U_n \cap S_E^G$ where $x_k^n \in \mathcal{M}$. Let $\alpha_k^n = \phi_1(x_k^n)$. Then $V_n^0 = \{\phi \in \mathcal{M}^*; \phi(x_k^n) = \alpha_k^n, k \leqq k_n\}$ satisfies $\phi_1 \in V_n^0 \cap S_E^G \subset V_n \cap S_E^G$. Let $\mathcal{J} = \{x_k^n - \alpha_k^n I; k, n \geqq 1\}$. Then

$$\phi_1 \in S_E^G \cap \mathcal{J}^0 = \bigcap_n V_n^0 \cap S_E^G \subset U \cap S_E^G.$$

Our Theorem 1 implies that $S_E^G \cap \mathcal{J}^0 \cap \{\mathcal{M}^* \sim \mathcal{M}_*\} \neq \emptyset$. (If $U = \mathcal{J}^0$ then clearly $S_E^G \cap \mathcal{J}^0 \cap \{\mathcal{M}^* \sim \mathcal{M}_*\} \neq \emptyset$.) Let now

$$\phi_0 \in S_E^G \cap \mathcal{J}^0 \cap \{\mathcal{M}^* \sim \mathcal{M}_*\} \subset S_E^G \cap U \cap \{\mathcal{M}^* \sim \mathcal{M}_*\}.$$

Since $\mathcal{M}$ is σ-finite and ϕ_0 is not normal there exists a sequence of pairwise disjoint projections p_n in $\mathcal{M}$ such that $d = \phi_0(\Sigma_1^\infty p_n) > \Sigma_1^\infty \phi_0(p_n)$ (see [23] p. 136 Cor. 3.11). Let $d_n = \phi_0(p_n)$ and S_N be the set of normal states on $\mathcal{M}$. Let $\mathcal{J}_1 = \mathcal{J} \cup \{p_n - d_n I; n \geqq 1\}$ and denote $A = w^* \operatorname{cl} S_N \cap S^G \cap \mathcal{J}_1^0$. Clearly $\phi_0 \in A$ and $A \cap w^* \operatorname{seq} \operatorname{cl} S_N = \emptyset$. This holds since $w^* \operatorname{seq} \operatorname{cl} S_N = S_N$ ([23] p. 148) and any $\phi \in S_N$ satisfies $\phi(\Sigma_1^\infty p_n) = \Sigma_1^\infty \phi(p_n)$. But then our Theorem 1.4 of [12] p. 158 (see also p. 171) implies that there exists a w^*-w^* continuous norm isomorphism into $t^*: l^{\infty *} \to \mathcal{M}^*$ such that $t^*\mathcal{F} \subset A$. Thus there is a $\delta > 0$ such that $\| t^*\phi \| \geqq \delta \| \phi \|$ for all ϕ. If $H = t^*(\beta N \sim N)$ then $\| \psi_1 - \psi_2 \| \geqq 2\delta$ if $\psi_1 \neq \psi_2$ are in H (since $\| \phi_1 - \phi_2 \| = 2$ if $\phi_1 \neq \phi_2$ are in $\beta N \sim N$), card $H = 2^c$ and H is a w^* perfect set. ∎

Remark. Let $\mathcal{J}$ be countable be such that $S_E^G \cap \mathcal{J}^0 \neq \emptyset$. Then our Theorem 2 only insures that $S_E^G \cap \mathcal{J}^0 \cap \{\mathcal{M}^* \sim \mathcal{M}_*\} \neq \emptyset$. It may though happen that $S^G \cap \mathcal{J}^0 \cap \mathcal{M}_*^\perp = \emptyset$ and *a fortiori* $S_E^G \cap \mathcal{J}^0 \cap \mathcal{M}_*^\perp = \emptyset$. In fact let

$\mathcal{M} = l^\infty$, $G = \{e\}$ be the trivial group. Let $\phi_1 \in (l^\infty)_* = l^1$ and $\phi_2 \in (l^\infty)_*^\perp$ be states on $\mathcal{M}$. Let $\phi_0 = \frac{1}{2}(\phi_1 + \phi_2)$ and $\alpha_n = \phi_0(\delta_n)$. Let $\mathcal{J} = \{\delta_n - \alpha_n I;\ n \geqq 1\}$. If E is the set of all pure states, then $S_E^G = S^G$ and $\phi_0 \in S^G \cap \mathcal{J}^0 \cap \{\mathcal{M}^* \sim \mathcal{M}_*\}$. Yet $S^G \cap \mathcal{J}^0 \cap \mathcal{M}_*^\perp = \varnothing$. Since any ϕ in this last set satisfies $\phi(\delta_n) = \alpha_n = \frac{1}{2}\phi_1(\delta_n)$ and $\Sigma_n \phi_1(\delta_n) = 1$ since $\phi_1 \in l^1$ is a state. Thus $\phi \notin (l^\infty)_*^\perp = \{\psi \in l^{\infty *};\ \psi = 0 \text{ on } c_0\}$.

We will need in what follows the following important result of Ghoussoub, Godefroy, Maurey, Schachermeyer, Haagerup [8].†

Lemma GGMSH. *Let D be a w^* compact convex subset of the dual of a C^* algebra (or of the dual of a Banach lattice not containing c_0). Then D has the RNP if and only if it has the WRNP.*

Proof. By Cor. VII. 5, p. 93 of [8], D has the RNP iff it has the W^* RNP since $D \subset A^*$, which is the predual of A^{**}. But by [8] p. 80, just preceding Theorem VI. 12, WRNP and W^* RNP coincide for w^* compact convex subsets of duals of Banach spaces. ■

Remark. If $G \subset \operatorname{Aut} \mathcal{M}$ is a group and E a set of pure states on $\mathcal{M}$, let $\hat{E} = \{\pi_\phi;\ \phi \in E\}/\sim$ where π_ϕ is the GNS representation induced by ϕ and $\sim$ denotes unitary equivalence. Then $\hat{g}\hat{E} = \hat{E}$ means that $\{\pi_\phi;\ \phi \in E\}/\sim\ = \{\pi_\phi \circ g;\ \phi \in E\}/\sim$ as in [13] p. 755.

We have now the following improvement of our Corollary 2 [13] p. 758:

Theorem 3. *Let $G \subset \operatorname{Aut} \mathcal{M}$ be a group and E a set of pure states on $\mathcal{M}$. Assume that for some countable $\mathcal{J} \subset \mathcal{M}$, $S_E^G \cap \mathcal{J}^0 \neq \varnothing$ and has the WRNP.*

(a) *If G is countable and amenable, then every subset $E_1 \subset w^* \operatorname{cl} E$ such that $G^* E_1 \subset E_1$ and $S_{E_1}^G \cap \mathcal{J}^0 \neq \varnothing$ contains a* finite *subset $E_0 \subset E_1$ such that $\hat{g}\hat{E}_0 = \hat{E}_0$ for all $g \in G$.*

(b) *If G is not amenable, then the action of G on $L^\infty(T^G, \lambda^G)$ is such that if $E = \Delta_{L^\infty}$, $\mathcal{J} = \{0\}$ then $S_E^G \cap \mathcal{J}^0 = S^G = \{\lambda^G\}$ (which is finite-dimensional a fortiori) has the WRNP, yet for each $\phi \in E$, $G^*\phi$ is infinite.*

Proof. Lemma GGMSH implies that $S_E^G \cap \mathcal{J}^0$ has in fact the RNP. The proof is then reduced to that of Cor. 2 on p. 758 of [13]. ■

† Many thanks are due to N. Ghoussoub for providing us with a preprint of [8].

REMARK. This theorem implies that RNP can be replaced by WRNP in our Cor. 2 and Cor. 3 on p. 758 of [13] and makes Cor. 4 of [13] superfluous. Furthermore, Lemma GGMSH shows that RNP can be replaced by WRNP in our Theorem 4 of [11] and makes Proposition 5 of [11] superfluous.

II. "Bigness" properties of $S^G \cap \mathcal{M}_*^\perp$

The main results of this section are Theorems 7 and 9.

LEMMA 4. *Let $\mathcal{M} \subset B(H)$ be an infinite-dimensional W^* algebra and $G \subset \mathrm{Aut}\,\mathcal{M}$ countable and amenable. Then there exist σ-finite projections q_n, $n = 0, 1, 2, \ldots$ in $\mathcal{M}$ and $\psi_0 \in S^G \cap \mathcal{M}_*^\perp$ such that: $gq_0 = q_0$ for all g in G, $q_n \uparrow q_0$ σ-strongly, $\psi_0(q_0) = 1$ and $\psi_0(q_n) = 0$ if $n \geqq 1$.*

PROOF. If $\mathcal{M}$ is not σ-finite, there is an infinite sequence $\{h_n\}$ such that $[\mathcal{M}'h_k] \perp [\mathcal{M}'h_j]$ if $k \neq j$, $n = 1, 2, 3, \ldots$ where $\mathcal{M}'$ is the commutant of $\mathcal{M}$ and $[\mathcal{M}'K]$ is the closed linear span of $\{mh;\, m \in \mathcal{M}', h \in K\}$ (see [23] p. 78). Let $H_0 = \{h_n\}_1^\infty$ and $p_0 \in \mathcal{M}$ the projection onto $[\mathcal{M}'H_0]$. Then $p_0\mathcal{M}p_0$ acting on $p_0H = [\mathcal{M}'H_0]$ is σ-finite, since the countable set H_0 is separating for $p_0\mathcal{M}p_0$ and by [23] p. 78 (3.19). Thus p_0 is a σ-finite projection by definition. Note that if $p_n \in \mathcal{M}$ is the projection onto $[\mathcal{M}'h_n]$ then $p_0 \geqq p_n$ and $p_k p_j = 0$ if $i \neq j$, $i, j \geqq 1$. Thus $\{p_n\}_1^\infty \subset p_0\mathcal{M}p_0$ and $p_0\mathcal{M}p_0$ is infinite dimensional. Now for each $g \in G$, $gp_0\mathcal{M}gp_0 = g(p_0\mathcal{M}p_0)$ is also a σ-finite since $g \in \mathrm{Aut}\,\mathcal{M}$. But, since G is countable, $q_0 = \sup\{gp_0;\, g \in G\}$ (denoted by $\{\vee gp_0;\, g \in G\}$ in [23] p. 290) is also a σ-finite projection in $\mathcal{M}$ (see [14] p. 380, ex. 5.7.45) which in addition is such that $gq_0 = q_0$ (since clearly $g_0q_0 = \sup\{g_0gp_0;\, g \in G\} = \sup\{gp_0;\, g \in G\} = q_0$). It follows that $q_0\mathcal{M}q_0$ is a σ-finite W^* algebra acting on q_0H which is infinite dimensional, since $q_0\mathcal{M}q_0 \supset g(p_0\mathcal{M}p_0)$ is infinite dimensional for each $g \in G$. Clearly $g(q_0\mathcal{M}q_0) = q_0\mathcal{M}q_0$ for all $g \in G$. Now if $\mathcal{N}$ is a W^* algebra such that $\dim \mathcal{N} = \infty$, then $\mathcal{N}^* = \mathcal{N}_* \oplus \mathcal{N}_*^\perp$ by [23] p. 127 and furthermore $\mathcal{N}_*^\perp \neq \{0\}$. If $\mathcal{N}_*^\perp = \{0\}$ then $\mathcal{N}^* = \mathcal{N}_*$ and $\mathcal{N}$ would be a reflexive W^* algebra and hence by S. Sakai's result in [20] would be finite dimensional. Let now ϕ be a singular ([23] p. 127) state on $\mathcal{N} = q_0\mathcal{M}q_0$ acting on q_0H. If $\mathcal{M}$ is σ-finite take $q_0 = I$ and $\mathcal{N} = \mathcal{M}$. G acts on $q_0\mathcal{M}q_0$. Hence $G^* = \{g^*;\, g \in G\}$ acts on $(q_0\mathcal{M}q_0)^*$. By the Markov–Kakutani–Day fixed point theorem there is some $\psi \in w^*\,\mathrm{cl}\,\mathrm{co}\{G^*\phi\}$ such that $G^*\psi = \psi$. But each $g^*\phi$ is clearly a singular state on $q_0\mathcal{M}q_0$, for example by Theorem 3.8 on p. 134 of [23], and ψ is in the w^* closure of a countable subset of $\mathrm{co}\, G^*\phi$ (all of whose elements are singular) since G^* is countable. Using now Prop. 5.8 on p. 154 of

[23] we get that ψ is a singular G^* invariant state on the σ-finite W^* algebra $q_0\mathcal{M}q_0$. Thus there is a net of projections $q_\alpha \uparrow q_0$, the identity of $q_0\mathcal{M}q_0$ being such that $\psi(q_\alpha)=0$; see [23] p. 154.

Let (by Takesaki [23] p. 78) w be a faithful normal positive state in $(q_0\mathcal{M}q_0)_*$. Then $w(q_0-q_\alpha)\to 0$. Choose q_{α_1} such that $w(q_0-q_{\alpha_1})<1$ and if q_{α_n} was chosen let $q_{\alpha_{n+1}} \geqq q_{\alpha_n}$ be such that $w(q_0-q_{\alpha_{n+1}})<1/n+1$. If $q_n=q_{\alpha_n}$ then $q_n \uparrow q_0$ ultrastrongly, q_n are σ-finite and $\psi(q_n)=0$ if $n \geqq 1$ while $\psi(q_0)=1$.

Let now $\psi_0 \in \mathcal{M}^*$ be defined by $\psi_0(m)=\psi(q_0mq_0)$. Then $g^*\psi_0(m)=\psi(q_0gmq_0)=\psi(q_0mq_0)=\psi_0(m)$ and also $\psi_0(m)=\psi(m)$ if $m\in q_0\mathcal{M}q_0$. Thus ψ_0 is a G^* invariant extension of ψ, which satisfies $\psi_0(q_n)=0$ and $\psi_0(1-q_0)=0$; thus $\psi_0(q_0)=1=\psi_0(I)$. Hence $\psi_0\in\mathcal{M}_*^\perp\cap S^G$ as required. (If $\psi_0=\psi_0^s+\psi_0^n$ with $\psi_0^s(\psi_0^n)$ the singular (normal) part of ψ_0 then $\psi_0((I-q_0)+q_k)=0$ implies $\psi_0^n(I-q_0+q_k)=0$. Hence $\psi_0^n(I)=0$, i.e. $\psi_0=\psi_0^s$.) ■

The following theorem is an improvement of Theorem 3.8 of Ching Chou [1] which in turn improves our Theorem 1, p. 117 of [10]. The main idea of the construction of the w^*-w^* continuous isometry $t^*: l^{\infty *}\to\mathcal{M}^*$ is due to Ching Chou [1]. The first and last parts of the proof necessarily differ from Chou's proof.

THEOREM 5. *Let $\mathcal{M}$ be an infinite-dimensional W^* algebra, $g_n:\mathcal{M}_*\to\mathcal{M}_*$ a sequence of bounded operators and $\mathcal{J}\subset\mathcal{M}$ countable. Let K be a convex set of normal states and $A=\{w^*\operatorname{cl}K\}\cap\{\phi\in\mathcal{M}^*;\ g_n^{**}\phi=0,\ \forall n\geqq 1\}\cap\mathcal{J}^0$. If $\varnothing\neq A\subset\mathcal{M}_*^\perp$ then there exists a w^*-w^* continuous positive* isometry *into* $t^*: l^{\infty *}\to\mathcal{M}^*$ *such that $t^*\mathcal{F}\subset A$.*

REMARKS. The key word in the above theorem is "*isometry*". Were we content to only have t^* to be a w^*-w^* positive norm *isomorphism* into (i.e. such that $c_1\|y\|\leqq\|t^*y\|\leqq c_2\|y\|$, for some $c_2>c_1>0$, for all y) we could have used our Theorem 1.4 in [12] p. 158 as done in Theorem 2.6, p. 171 of [12]. (We note that $A\subset\mathcal{M}_*^\perp$ implies that $A\subset w^*\operatorname{cl}K\sim w^*\operatorname{seq}\operatorname{cl}K$, since $w^*\operatorname{seq}\operatorname{cl}K\subset\mathcal{M}_*$, by [23] p. 148).

PROOF. Let $\phi_0\in A\subset\mathcal{M}_*^\perp$. Then for any normal state ϕ one has $\|\phi-\phi_0\|=2$. Indeed, if Z_0 is the central projection in $\mathcal{M}^{**}$ for which $\mathcal{M}_*=\mathcal{M}^*Z_0$ (see [23] p. 126) then $\phi Z_0=\phi$ and $\phi_0Z_0=0$. Thus

$$\|\phi-\phi_0\|=\|(\phi-\phi_0)Z_0\|+\|(\phi-\phi_0)(1-Z_0)\|$$
$$=\|\phi Z_0\|+\|\phi_0(1-Z_0)\|=2$$

by [23] Theorem 2.4, p. 127. If now ϕ_α is any net in K such that $\phi_\alpha \to \phi_0$ in w^* then $\lim_\alpha \| \phi_\alpha - \phi \| = 2$ for any normal state ϕ since $\underline{\lim} \| \phi_\alpha - \phi \| \geqq \| \phi_0 - \phi \| = 2$. The above replaces the need for Lemma 3.1 of Ching Chou [1] p. 214. Let now ψ_α a net in K be such that $\psi_\alpha \to \phi_0$ in w^*. Then $w^* \lim_\alpha g_n^{**}\psi_\alpha = g_n^{**}\phi_0 = 0$ and $\langle x, \psi_\alpha \rangle \to 0$ for all x in $\mathscr{J}$. Using now our extension of an argument of Namioka [9] p. 18 (as in [12] or [1]) there is a net ϕ_α of convex combinations of $\{\psi_\alpha\}$ such that $\phi_\alpha \to \phi_0$ in w^*, $\| g_n \phi_\alpha \| \to 0$ $\forall n$ and $\langle x, \phi_\alpha \rangle \to 0$ for all x in $\mathscr{J}$. As above $\| \phi_\alpha - \phi \| \to 2$ for all normal states ϕ. Write $\mathscr{J} = \{x_n; n = 1, 2, \ldots\}$. Choose now a sequence ϕ_{α_n} such that $\| g_i \phi_{\alpha_n} \| < 1/n$ and $| \langle x_i, \phi_{\alpha_n} \rangle | < 1/n$ for all $1 \leqq i \leqq n$ and denote ϕ_{α_n} by ϕ_n. Now as in [12] p. 154 any w^* limit ψ of a subnet $\{\phi_{n_\beta}\}$ of the sequence ϕ_n ncessarily satisfies $g_n^{**}\psi = 0$ and $\langle \psi, x_n \rangle = 0$ $\forall n \geqq 1$ and hence $\psi \in A$. But $A \subset \mathscr{M}_*^\perp$ by assumption. Thus $\| \psi - \phi \| = 2$ for any normal state ϕ and the same reasoning as above shows that $\| \phi_n - \phi \| \to 2$ for any normal state ϕ. We can apply now Ching Chou's powerful Theorem 2.4 on p. 212 of [1] and get that there is a subsequence ϕ_{n_j} of ϕ_n and normal states of ψ_j such that $\| \phi_{n_j} - \psi_j \| \leqq 1/j$ and the ψ_j are mutually orthogonal, i.e. the support projections $p_j \in \mathscr{M}$ of ψ_j (see [23] p. 134) satisfy $p_j p_k = 0 = p_k p_j$ if $j \neq k$. Thus $\{\psi_j\}$ form a *canonical* l_1 *basis* in $\mathscr{M}_*$, i.e. any linear combination of ψ_j satisfies $\| \Sigma_1^k \alpha_j \psi_j \| = \Sigma_1^k |\alpha_j|$. In fact if $x = \Sigma_1^k(\bar{\alpha}_j / |\alpha_j|) p_j$ then $\| x \| = 1$ and $\langle \Sigma_1^k \alpha_j \psi_j, x \rangle = \Sigma_1^k |\alpha_j|$. It is now readily checked that $\| g_n \psi_j \| \to 0$ and $\langle x_n, \psi_j \rangle \to 0$ for each fixed n. Furthermore the operator $i : l_1 \to \mathscr{M}_*$ given by $i : \{\alpha_j\} \to \Sigma_1^\infty \alpha_j \psi_j$ is an *into* isometry, i.e. $\| i\{\alpha_j\}_1^\infty \| = \Sigma_1^\infty |\alpha_j|$. Also $i\delta_k = \psi_k$, and $i \geqq 0$, i.e. if $\alpha_j \geqq 0$ for all j then $i\{\alpha_j\} \geqq 0$ in $\mathscr{M}_*$. But then if $b = \{b_n\} \in l^\infty$, define $\hat{b} \in \mathscr{M}$ to be any norm preserving extension of the linear functional $\hat{b}_0$ on $i(l^1)$ given by: $\langle \hat{b}_0, \Sigma_1^\infty \alpha_j \psi_j \rangle = \Sigma_1^\infty b_j \alpha_j$. Note that $\| \hat{b} \| = \| \hat{b}_0 \| = \sup_n |b_n| = \| b \|$.

Thus $i^*\hat{b} = b$ and $\| \hat{b} \| = \| b \|$. But then since $\| i^* \| \leqq 1$, $\| i^{**}\phi \| = \| \phi \|$ for any $\phi \in l^{\infty *}$ and $i^{**} : l^{\infty *} \to \mathscr{M}^*$ is an isometry into, which is positive since so is $i : l^1 \to \mathscr{M}_*$.

We show now that $i^{**}\theta \in A$ if $\theta \in \mathscr{F}$. Clearly $i\delta_k = \psi_k$ and if $x \in \mathscr{M}$ then for fixed n we have $\langle i^{**}g_n^* x, \delta_k \rangle = \langle x, g_n \psi_k \rangle \to 0$ if $k \to \infty$ and also $\langle i^* x_n, \delta_k \rangle = \langle x_n, \psi_k \rangle \to 0$ if $k \to \infty$.

Hence for all n, $i^* g_n^*(\mathscr{M}) \subset c_0$ and $i^*\mathscr{F} \subset c_0$. Let now $\theta \in \mathscr{F}$ be fixed. Then $\theta = 0$ on c_0. Hence

$$\langle g_n^{**} i^{**}\theta, x \rangle = \langle \theta, i^* g_n^* x \rangle = 0 \quad \text{and} \quad \langle i^{**}\theta, x_n \rangle = \langle \theta, i^* x_n \rangle = 0.$$

We still have to show that $i^{**}\theta \in w^* \operatorname{cl} K$ to finish the proof. Here again we differ from Ching Chou's proof [1] p. 217.

Since $\langle \theta, \delta_k \rangle = 0$ for all k we have for all $j \geqq 1$ that $\theta \in w^*\,\mathrm{cl}\,\mathrm{co}\{\delta_k; k \geqq j\}$ thus $i^{**}\theta \in w^*\,\mathrm{cl}\,\mathrm{co}\{\psi_k; k \geqq j\}$. We note that if j is fixed then $\| \phi_{n_k} - \psi_k \| \leqq 1/j$, if $k \geqq j$. Hence if $\mu = \sum_{k=0}^{l} \alpha_k \psi_{k+j} \in \mathrm{co}\{\psi_n; n \geqq j\}$ then $\mu' = \sum_{k=0}^{l} \alpha_k \phi_{n_{k+j}} \in K$ will satisfy $\| \mu - \mu' \| \leqq 1/j$.

For every finite set $F \subset \mathcal{M}$ and $n \geqq 1, j \geqq 1$ there is some $v \in \mathrm{co}\{\delta_k; k \geqq j\}$ such that $|\langle i^{**}(v - \theta), x \rangle| < 1/n$ for all x in F. Let D be the directed set $\{\beta = (F, j, n),\ F \subset \mathcal{M}$ finite, $n, j \geqq 1\}$ with $(F', j', n') \geqq (F, j, n)$ iff $F \subset F'$, $n' \geqq n$ and $j' \geqq j$. Then it is readily seen that there is a net on D, $\{v_\beta; \beta \in D\} \subset \mathrm{co}\{\delta_k; k \geqq 1\}$ such that if $\beta = (F, j, n)$ then $v_\beta \in \mathrm{co}\{\delta_k; k \geqq j\}$ and $|\langle i^{**}(v_\beta - \theta), x \rangle| < 1/n$ if $x \in F$ and such that $v_\beta \to \theta$ in w^*.

But then let $\mu_\beta = i^{**} v_\beta$ and μ'_β be the net containing ϕ_{n_j} at each place where ψ_j appears in $i^{**} v_\beta = \mu_\beta$. Then $\mu'_\beta \in K$ and $\| \mu_\beta - \mu'_\beta \| \leqq 1/j$ if $\beta \geqq \beta_0 = \{F_0, k, j\}$ for any k and finite $F_0 \subset \mathcal{M}$.

Let now $\varepsilon > 0$ and $F_0 \subset \mathcal{M}$ be finite. Let $\beta_0 = (F_0, k_0, j_0)$ be such that $|\langle i^{**}(v_\beta - \theta), x \rangle| < 1/j_0 < \varepsilon/2$, $\forall x \in F_0$ if $\beta \geqq \beta_0$ and furthermore $\| x \| / j_0 < \varepsilon/2$ $\forall x \in F_0$. Then $\forall x \in F_0$

$$|\langle \mu'_\beta - i^{**}\theta, x \rangle| \leqq |\langle \mu'_\beta - \mu_\beta, x \rangle| + |\langle i^{**}(v_\beta - \theta), x \rangle|$$

$$< (\| x \| / j_0) + \varepsilon/2 < \varepsilon.$$

This shows that $i^{**}\theta \in w^*\,\mathrm{cl}\,K$. Thus $i^{**}\mathcal{F} \subset A$. Denote $t = i^*$. ∎

COROLLARY 6. *Assumptions as in Theorem 5 except that $\emptyset \neq A \subset \mathcal{M}_*^\perp$ is replaced by $A \cap \mathcal{M}_*^\perp \neq \emptyset$ and $\mathcal{M}$ is σ-finite. Then there exists a w^*–w^* continuous positive isometry into $t^*: l^{\infty *} \to \mathcal{M}^*$ such that $t^*\mathcal{F} \subset A \cap \mathcal{M}_*^\perp$.*

PROOF. Let $\phi_0 \in A \cap \mathcal{M}_*^\perp$ and choose a sequence of projections $p_n \uparrow I$, ultrastrongly in $\mathcal{M}$, such that $\phi_0(p_n) = 0$ for all n (see Takesaki [23] p. 154 and Theorem 3.8 p. 134). Let $\mathcal{J}_1 = \mathcal{J} \cup \{p_n\}$. Then:

$$A_1 = \{w^*\,\mathrm{cl}\,K\} \cap \{\phi \in \mathcal{M}^*; g_n^{**}\phi = 0\ \forall n \geqq 1\} \cap \mathcal{J}_1^0 \subset \mathcal{M}_*^\perp \quad \text{and} \quad \phi_0 \in A_1.$$

This is the case since if $\phi \in A_1$ and $\phi = \phi^n + \phi^s$ with ϕ^n (ϕ^s) the normal (singular) part of ϕ, then $\phi(p_k) = 0$ implies $\phi^n(p_k) = 0$ for all k. Thus $\phi^n(I) = 0$ and $\phi = \phi^s$ is singular. Apply now Theorem 5 to finish the proof. ∎

THEOREM 7. *Let $\mathcal{M}$ be an infinite-dimensional W^* algebra, $G \subset \mathrm{Aut}\,\mathcal{M}$ a countable* amenable *group. Then there is a sequence of σ-finite projections $\{q_n; n \geqq 0\}$ such that $q_n \uparrow q_0$ σ-strongly and $gq_0 = q_0$ for all g in G, and a positive w^*–w^* continuous* isometry *into $t^*: l^{\infty *} \to \mathcal{M}^*$ such that*

$$t^*\mathscr{F} \subset S^G \cap \mathscr{M}_*^\perp \cap P^0, \qquad \text{where } P = \{q_n; n \geqq 1\} \cup \{I - q_0\}.$$

PROOF. Choose $q_n \uparrow q_0$ and $\psi_0 \in S^G \cap \mathscr{M}_*^\perp$ as in Lemma 4. Let K be any convex set of normal states such that $\psi_0 \in w^*\text{ cl } K$ (the set S_N of all normal states is such). Let $P = \{q_n; n \geqq 1\} \cup \{I - q_0\}$. Then

$$A = \{w^* \text{cl } K\} \cap \{\phi \in \mathscr{M}^*; G^*\phi = \phi\} \cap P^0 \subset \mathscr{M}_*^\perp \quad \text{and} \quad \psi_0 \in A.$$

Since any $g \in \text{Aut } \mathscr{M}$ is ultraweakly continuous ([23] p. 135) $g^*\mathscr{M}_* \subset \mathscr{M}_*$ for all g, hence for each g there is some $g_1 : \mathscr{M}_* \to \mathscr{M}_*$ such that $g_1^* = g$. Thus a direct application of Theorem 5 with $\{g_n^{**}\}$ replaced by $\{(g_1 - I)^{**}; g \in G\}$ finishes the proof. ∎

Since $\beta N \sim N$ is a w^* perfect set, as is well known, we have

COROLLARY 8. *The set $H = t^*(\beta N \sim N) \subset S^G \cap \mathscr{M}_*^\perp \cap P^0$ is a w^* compact perfect subset of cardinality 2^c, such that $\| \phi_1 - \phi_2 \| = 2$ for any $\phi_1, \phi_2 \in H$ such that $\phi_1 \neq \phi_2$.*

Theorem 7 and Corollary 8 improve substantially the following result of Ching Chou [1] p. 649:

COROLLARY (Ching Chou). *Assume that G is a countable amenable group acting ergodically as measure preserving maps on a nonatomic probability space $(X, \mathscr{B}, p)$. Then there exists a positive w^*-w^* continuous isometry into $t^* : l^{\infty *} \to L^\infty(X)^*$ such that $t^*\mathscr{F} \subset S^G$.*

We note that the action of G on our W^* algebra need not be trace preserving ($\mathscr{M}$ need not be abelian or admit any faithful trace at all) or ergodic, and $\mathscr{M}$ may contain atoms. Moreover $t^*\mathscr{F}$ contains only singular elements ψ such that $\psi(q_n) = 0$ if $n \geqq 1$, $\psi(q_0) = 1$. (Hence if $r_n = q_n + (I - q_0)$, $n \geqq 1$ then r_n are projections such that $r_n \uparrow I$, σ strongly, and $\psi(r_n) = 0$ for all $\psi \in t^*\mathscr{F}$, $n \geqq 1$. This assertion is related to Theorem 1.4 (1)$\Leftrightarrow$(3) of Rosenblatt [17] p. 628.)

REMARK (1). The following is a result in K. Schmidt [21] p. 227.

THEOREM. *Let G be a countable group and $(X, \mathscr{B}, \mu)$ be a standard nonatomic probability space TFAE: (1) No ergodic measure preserving action of G on $(X, \mathscr{B}, \mu)$ is strongly ergodic. (2) No ergodic measure preserving action of G on $(X, \mathscr{B}, \mu)$ has a unique G invariant state on $L^\infty(X, \mu)$. (3) G is amenable.*

Our theorem implies that amenable countable groups have a much stronger property than (2), namely: Every action (null set preserving but not necessarily measure preserving) of G on any measure space $(X, \mathscr{B}, \mu)$ such that $\mathscr{M} = L^\infty(X, \mu)$ is not finite dimensional satisfies card $S^G \cap \mathscr{M}_*^\perp = 2^c$. Furthermore, the abelian $\mathscr{M} = L^\infty(X)$ can be replaced by *any* W^* algebra with dim $\mathscr{M} = \infty$.

REMARK (2). Let $\mathscr{M}$ be a finite W^* algebra with faithful normal trace τ such that $\tau(I) = 1$. For a fixed τ let $\| x \|_2 = \tau(x^*x)^{1/2}$ if $x \in \mathscr{M}$. Let $G \subset \mathrm{Aut}\,\mathscr{M}$ preserve τ and extend the action of each $g \in G$ to the canonical Hilbert space $L^2(\mathscr{M}, \tau)$. M. Choda [4] defines the action of G on $\mathscr{M}$ to be s-strongly ergodic if, whenever $\xi_n \in L^2(\mathscr{M}, \tau)$ are unit vectors such that $\| \xi_n - g\xi_n \|_2 \to 0$ for each g in G, it follows that $\| \xi_n - \tau(\xi_n)1 \|_2 \to 0$. She proves in [4], among other results, that "The action of G on $\mathscr{M}$ is s-strongly ergodic if and only if τ is the unique G invariant state on $\mathscr{M}$."

One can hence call the action of G on an *arbitrary* W^* algebra $\mathscr{M}$ to be s-strongly ergodic if it admits a unique G invariant state. Then one gets that the action of countable amenable groups on any W^* algebra $\mathscr{M}$ is never s-strongly ergodic. In fact, moreover, dim $S^G \cap \mathscr{M}_*^\perp \geqq 2^c$ (if dim $\mathscr{M} = \infty$).

For countable groups with property T (for ex. $\mathrm{Sl}(nZ)$, $n \geqq 3$) the situation is strikingly different as the following result of K. Schmidt [21] and A. Connes and B. Weiss [5] shows: "A countable group G has property T, iff any measure preserving ergodic action on any standard probability space is strongly ergodic".

The main result of Ching Chou in [1] is

THEOREM. *Let G be a countable group acting ergodically as measure preserving maps on the nonatomic probability space $(X, \mathscr{B}, p)$. If the set S^G of G invariant states on $\mathscr{M} = L^\infty(X, p)$ satisfies $\{ p \} \subsetneqq S^G$, then there is a positive w^*–w^* continuous isometry into $t^* : l^{\infty^*} \to \mathscr{M}^*$ such that $t^*\mathscr{F} \subset S^G$. In particular*, card $S^G \geqq 2^c$.

We note that G is not necessarily amenable here and that $\mathscr{M}$ is a finite W^* algebra ([16] p. 166). Furthermore, due to ergodicity, $\{ p \}$ is the unique G invariant *normal* state on $\mathscr{M} = L^\infty(X)$. It is hence assumed above that there is some $\psi \in S^G$ which is not normal.

Ching Chou's result is thus substantially improved in

THEOREM 9. *Let $G \subset \mathrm{Aut}\,\mathscr{M}$ be any countable group acting on the σ-finite W^* algebra $\mathscr{M}$. Assume that there is some $\phi_0 \in S^G$ which is not normal. Then there is a sequence of projections $q_n \uparrow I$, σ-strongly, and a positive w^*–w^**

continuous isometry $t^*: l^{\infty *} \to \mathcal{M}^*$ *such that* $t^*\mathcal{F} \subset S^G \cap \mathcal{M}_*^\perp \cap P^0$ *where* $P = \{q_n\}$.

REMARK. Ching Chou's proof in [1] p. 648 relies on J. Rosenblatt's Theorem A in which essential use is made of the fact that G preserves the finite measure p (see [17] p. 627). In our case $\mathcal{M}$ need not admit any faithful, finite (or σ-finite) trace at all and even if $\mathcal{M}$ admits one, it need not be preserved by G.

PROOF. Let Z_0 be the central projection in $\mathcal{M}^{**}$ such that $\mathcal{M}_* = \mathcal{M}^*Z_0$ and $\mathcal{M}_*^\perp = \mathcal{M}^*(1 - Z_0)$, see Takesaki [23] p. 126, and $\mathcal{M}^* = \mathcal{M}_* \oplus \mathcal{M}_*^\perp$ (an l^1 direct sum). Then $\eta_0 = (1 - Z_0)\,\phi_0 \neq 0$ and $\eta_0 \in \mathcal{M}_*^\perp$ is positive (since ϕ_0 is not normal and Z_0 is central). But $g^*\eta_0 = \eta_0$ for all g in G. To show this one notes that, since $g \in \operatorname{Aut} \mathcal{M}$, g is ultraweakly continuous ([23] p. 135), hence $g^*\mathcal{M}_* \subset \mathcal{M}_*$. Also, if $\phi \in \mathcal{M}_*^\perp$ is positive, then $g^*\phi \in \mathcal{M}_*^\perp$. Since if $p \neq 0$ is a projection in $\mathcal{M}$ then so is gp, and there is a projection $0 \neq p_1 \leqq gp$ such that $\phi(p_1) = 0$ (see [23] p. 134, Theorem 3.8), hence $g^*\phi(g^{-1}p_1) = \phi(p_1) = 0$ and $0 \neq g^{-1}p_1 \leqq p$ since g is an automorphism. Hence again by [23] p. 134, Theorem 3.8, $g^*\phi \in \mathcal{M}_*^\perp$. It follows that

$$\phi_0 = \phi_0 Z_0 + \phi_0(I - Z_0) = g^*\phi_0 = g^*(\phi_0 Z_0) + g^*(\phi_0(I - Z_0)).$$

Since $\mathcal{M}^* = \mathcal{M}_* \oplus \mathcal{M}_*^\perp$ is a direct sum, $g^*\eta_0 = g^*(\phi_0(1 - Z_0)) = \phi_0(1 - Z_0) = \eta_0$ for all g. Thus $\eta_0 \in \mathcal{M}_*^\perp$. It follows that $\psi_0 = \eta_0(I)^{-1}\eta_0 \in \mathcal{M}_*^\perp \cap S^G$. Now $\mathcal{M}$ is σ-finite ([23] p. 78) hence, as done after Corollary 8, there is a sequence of projections $q_n \uparrow I$, σ-strongly, such that $\psi_0(q_n) = 0$ if $n \geqq 1$. Let now $K = S_N$, the set of normal states on $\mathcal{M}$. Then

$$\psi_0 \in A = \{w^* \operatorname{cl} K\} \cap \{\psi \in \mathcal{M}^*; g^*\psi = \psi \text{ for } g \in G\} \cap P^0 \subset \mathcal{M}_*^\perp$$

by the same argument as at the end of Lemma 4. We apply now Theorem 5 and get that there is a w^*-w^* continuous positive into isometry $t^*: l^{\infty *} \to \mathcal{M}^*$ such that $t^*\mathcal{F} \subset S^G \cap \mathcal{M}_*^\perp \cap P^0$. ∎

COROLLARY 10. *The set* $H = t^*(\beta N \sim N) \subset S^G \cap \mathcal{M}_*^\perp \cap P^0$ *is a* w^* *compact perfect set such that* $\operatorname{card} H = 2^c$ *and* $\|\phi_1 - \phi_2\| = 2$ *if* $\phi_1, \phi_2 \in H$ *and* $\phi_1 \neq \phi_2$.

REMARKS. What can one say about the cardinality of S^G in case G is an uncountable amenable group? In this regard the following result of Z. Yang [24] is of much interest:

THEOREM. *Assuming the continuum hypothesis, there exists a locally finite (hence amenable) group G of cardinality c acting on a countable set X such that there exists a unique G invariant state on $\mathcal{M} = l^\infty(X)$.*

Thus any attempt to generalize the theorems of this paper to uncountable amenable groups G cannot work.

REFERENCES

1. Ching Chou, *Topological invariant means on the Von Neumann algebra* VN(G), Trans. Am. Math. Soc. **273** (1982), 207–229.
2. Ching Chou, *Ergodic group actions with nonunique invariant means*, Proc. Am. Math. Soc. **100** (1987), 647–650.
3. M. Choda, *Normality of invariant states under strongly ergodic actions*, Math. Japonicae **27** (1982), 293–300.
4. M. Choda, *Effect of inner amenability on strong ergodicity*, Math. Japonicae **28** (1983), 109–115.
5. A. Connes and B. Weiss, *Property T and asymptotically invariant sequences*, Isr. J. Math. **37** (1980), 209–210.
6. J. Cuntz, *On the continuity of seminorms on operator algebras*, Math. Ann. **220** (1976), 171–183.
7. M. M. Day, *Amenable semigroups*, Illinois J. Math. **1** (1957), 509–544.
8. N. Ghoussoub, G. Godefroy, B. Maurey and W. Schachermeyer, *Some topological and geometrical structures in Banach spaces*, Memoirs Am. Math. Soc. No. 378 (1987).
9. E. E. Granirer, *Exposed points of convex sets and weak sequential convergence*, Memoirs Am. Math. Soc. No. 123 (1972).
10. E. E. Granirer, *Properties of the set of topologically invariant means on P. Eymard's W*-algebra* VN(G), Indag. Math. **36** (1974), 116–121.
11. E. E. Granirer, *Geometric and topological properties of certain w*-compact convex sets which arise from the study of invariant means*. Canad. J. Math. **37** (1985), 107–121.
12. E. E. Granirer, *Geometric and topological properties of certain w*-compact convex subsets of double duals of Banach spaces which arise from the study of invariant means*, Illinois J. Math. **30** (1986), 148–174.
13. E. E. Granirer, *A strong containment property for discrete amenable groups of automorphisms on W*-algebra*, Trans. Am. Math. Soc. **297** (1986), 753–761.
14. R. V. Kadison and J. R. Ringrose, *Fundamentals of the Theory of Operator Algebras*, Vol. I, Academic Press, New York, 1983.
15. V. Losert and H. Rindler, *Almost invariant sets*, Bull. London Math. Soc. **13** (1981), 145–148.
16. G. K. Pedersen, *C*-algebras and their Automorphism Groups*, Academic Press, New York, 1979.
17. J. Rosenblatt, *Uniqueness of invariant means for measure preserving transformations*, Trans. Am. Math. Soc. **265** (1981), 623–636.
18. J. Rosenblatt and M. Talagrand, *Different types of invariant means*, J. London Math. Soc. **24** (1981), 525–536.
19. E. Saab, *Some characterisations of weak Radon-Nikodym sets*, Proc. Am. Math. Soc. **86** (1982), 307–311.
20. Sakai, *Weakly compact operators on operator algebras*, Pacific J. Math. **14** (1964), 659–664.

21. K. Schmidt, *Amenability property T strong ergodicity and invariant means for ergodic group actions*, Ergodic Theory and Dynamical Systems **1** (1981), 223–236.

22. C. Stegall, *The Radon–Nikodym property in conjugate Banach spaces II*, Trans. Am. Math. Soc. **264** (1981), 507–519.

23. M. Takesaki, *Theory of Operator Algebras I*, Springer-Verlag, Berlin–New York, 1979.

24. Zhuocheng Yang, *Action of amenable groups and uniqueness of invariant means*, preprint.